COLLAPSE OF AN ANNULAR NON-NEWTONIAN FLUID FILM IN AXIAL IMPACT

A. V. Dubovik, A. A. Denisaev,
and V. K. Bobolev

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A system of hydrodynamical equations is derived, describing the collapse of a thin annular film of non-Newtonian fluid in axial impact. Exact solutions are obtained and the cavity collapse curves calculated for certain special cases.

The need to investigate the behavior of a thin pseudoplastic interlayer between solid impacting surfaces arises in a number of technological problems connected with the theory of pressure treatment of plastics, hydrodynamic lubrication theory, etc.

The problem of a non-Newtonian squeeze film in axial impact has been solved earlier [1, 2]. The presence of a gas-filled cavity in the film complicates the solution of the problem due to the necessity of accounting for radial fluid flow not only outward, but also toward the impact symmetry axis. The same consideration distinguishes the present case from the collapse of a spherically symmetric cavity in an unbounded fluid volume when the pressure is increased to infinity.

We recall that the components of the stress deviator are related to the components of the strain-rate tensor as follows for a non-Newtonian fluid obeying a rheological power law [3]:

$$s_{ij} = 2m\,|2d_{ij}d_{ij}|^{\frac{n-1}{2}}\,d_{ij}, \tag{1}$$

where m and n are empirical constants of the material, which depend, in general, on the pressure and temperature, though n only very weakly so. For pseudoplastic materials we have $0 < n < 1$. In the limit $n = 0$ or $n = 1$ we obtain analogous relations from (1) for an ideal rigid plastic body and a normal (Newtonian) fluid, respectively.

In the axisymmetric squeeze flow of a thin ($\delta \ll R$) film we have $u \gg v$ and $\partial u/\partial z \gg \partial u/\partial r$. These conditions reduce (1) to the following simple relation between the tangential stress and shear rate:

$$\tau_{rz} = m\left|\frac{\partial u}{\partial z}\right|^n \operatorname{sign}\frac{\partial u}{\partial z}. \tag{2}$$

It follows from (2), in particular, that the apparent viscosity for pseudoplastic materials

$$\mu_a = m\left|\frac{\partial u}{\partial z}\right|^{n-1}$$

decreases with the shear rate.

We consider an axial impact with velocity $w_0 < 0$ against an incompressible non-Newtonian fluid film occupying a narrow gap between the plane-parallel surfaces of a striker and anvil of equal radius R.

At the center of the film, aligned with the axis of the striker, is a cylindrical cavity of radius r_0 and height δ_0 equal to the film thickness. The cavity is filled with an ideal gas with initial pressure p_a.

We place the origin of a cylindrical coordinate system at the center of the cavity on the surface of the anvil.

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For small Reynolds numbers $\mathrm{Re} \sim \rho_0 u\delta/\mu_a$, $u \sim wR/\delta$ the motion of the fluid is described by the system of equations of lubrication theory; in the axisymmetric case for $\delta_0 \ll R$ this system takes the form

$$\frac{\partial p}{\partial r} = m\frac{\partial}{\partial z}\left(\left|\frac{\partial u}{\partial z}\right|^{n-1}\frac{\partial u}{\partial z}\right); \quad \frac{\partial p}{\partial z} = 0;$$
$$\frac{1}{r}\cdot\frac{\partial ur}{\partial r} + \frac{\partial v}{\partial z} = 0. \tag{3}$$

The boundary conditions for (3) include the following: no-slip conditions between the fluid and the contact surfaces:

$$u(r,\ 0) = u(r,\ \delta) = 0;$$
$$v(r,\ 0) = 0; \quad v(r,\ \delta) = w; \tag{4}$$

zero net tangential stress at the boundary of the cavity; and equality between the normal stresses and pressure in the gas at the boundary. The latter conditions can with acceptable accuracy be replaced by the following approximate pressure conditions:

$$p(R,\ \delta) = p_a; \quad p(r_+,\ \delta) = p_b. \tag{5}$$

It may be assumed that the compression of the gas in the cavity is adiabatic and that the radial velocity $\dot{r}_+$ of its wall is much smaller than the velocity of sound. Then the pressure in the gas is uniform and equal to

$$p_b = p_a\left(\frac{r_0^2}{r_+^2}\cdot\frac{\delta_0}{\delta}\right)^{\gamma}. \tag{6}$$

To close the system (3)-(6) we must determine the deceleration law of the striker. The velocity w of the contact surface coincides with the velocity of the center of gravity of the striker only if the latter is absolutely nondeformable. If $K^{-1} = \sum\limits_{i=1}^{N} K_i^{-1}$ is the stiffness of the elements of the load system (N is the number of elements), then the displacement of the center of gravity of the striker is composed of the decrease in thickness of the film and the elastic deformation of the system. Applying Newton's law, we write the equation of motion of the striker:

$$M\frac{d^2}{dt^2}\left(\delta - \delta_0 - \frac{\langle p\rangle S}{K}\right) = \langle p\rangle S,$$
$$S = \pi R^2; \quad \langle p\rangle = p_b\left(\frac{r_+}{R}\right)^2 + \frac{2}{R^2}\int\limits_{r_+}^{R} prdr. \tag{7}$$

It follows from the equations of motion (3) and the boundary conditions (4) that

$$u = f(r,\ \delta)\left(1 - |1 - \eta|^{\frac{n+1}{n}}\right); \quad \eta = 2z/\delta. \tag{8}$$

Applying the equation of continuity and the boundary condition $v(r,\delta) = w$, we find

$$f = -\frac{2n+1}{n+1}\cdot\frac{wr}{2\delta} + \frac{c}{r},$$
$$v = \frac{w}{2(n+1)}\left[(2n+1)\eta - n\left(1 \pm |1-\eta|^{\frac{2n+1}{n}}\right)\right]. \tag{9}$$

The minus sign is chosen for $0 \le \eta \le 1$ (lower half of the film), and the plus sign for $1 \le \eta \le 2$ (upper half). Averaging u with respect to η, we satisfy the condition at the boundary of the cavity

$$\dot{r}_+ = \frac{1}{2}\int\limits_0^2 u(r_+,\ \delta)\,d\eta$$

and calculate

$$c = \frac{2n+1}{n+1}\,r_+\left(\dot{r}_+ + \frac{wr_+}{2\delta}\right).$$

We determine the position of the neutral line r_* from the condition $f = 0$:

$$r_* = r_+ (1 + 2\delta \dot{r}_+ / w r_+)^{1/2}. \tag{10}$$

We introduce the dimensionless variables $\xi = r_+/R$, $x = \delta_0/\delta$, $y = w/w_0$, $a = r_*/R$, $\rho = r/R$, $\tau = t/t_0$, $P = p/p_0$ and the parameters $t_0 = \delta_0/|w_0|$, $b = r_0/R$, $P_a = p_a/p_0$, $P_b = p_b/p_0$,

$$p_0 = 2m \left(\frac{2n+1}{n} \right)^n \frac{|w_0|^n R^{n+1}}{\delta_0^{2n+1}}.$$

Substituting (8) and (9) into the equation of motion (3) and integrating it subject to condition (5), we determine the pressure profile:

$$P = P_a \left(\frac{b^2 x}{\xi^2} \right)^\nu - g \int_\xi^\rho |\psi|^{n-1} \psi d\rho,$$
$$\psi = \rho - a^2/\rho; \quad g = x^{2n+1} y^n. \tag{11}$$

We obtain an equation for the cavity radius from (11), using the boundary condition

$$P(1, x) = P_a$$

and transforming to dimensionless variables in Eq. (10) and in the expression $w = d\delta/dt$:

$$\int_\xi^1 |\psi|^{n-1} \psi d\rho = \frac{P_a}{g} \left[\left(\frac{b^2 x}{\xi^2} \right)^\nu - 1 \right],$$
$$\frac{d\xi}{d\tau} = -\frac{xy\xi}{2} \left(\frac{a^2}{\xi^2} - 1 \right), \tag{12}$$
$$\frac{dx}{d\tau} = yx^2.$$

Substituting (11) into (7) and denoting $\alpha = p_0 S / K\delta_0$, $\beta = p_0 S \delta_0 / M w_0^2$, we obtain the striker deceleration equation:

$$\frac{dy}{d\tau} + \alpha \frac{d^2\Pi}{d\tau^2} + \beta\Pi = 0,$$
$$\Pi = \langle p \rangle / p_0 = P_a \xi^2 \left(\frac{b^2 x}{\xi^2} \right)^\nu + 2 \int_\xi^1 P\rho d\rho. \tag{13}$$

The motion of the fluid in the gap between the impacting surfaces causes it to heat up due to viscous dissipation of the mechanical energy input. If the Péclet number $Pe \sim \rho_0 c_p |w| \delta / \lambda \gg 1$, then the heat-flow equation is written

$$\rho_0 c_p \frac{dT}{dt} = m \left| \frac{\partial u}{\partial z} \right|^{n+1}. \tag{14}$$

Here T is interpreted as the temperature increase (heating) of the fluid.

It follows from (9) that the flow velocity gradient is a maximum at the points $\eta = 0.2$ and the function $|f|$ attains its largest values at $r = R$, r_+. Inasmuch as $(1 - \sqrt{\xi})^2 > 0$, the quantity

$$|f(\xi)|/|f(1)| = (a^2/\xi - \xi)/(1 - a^2) > 0,$$

and so the heating of the fluid at the boundary with the cavity is greater than the heating at the edge of the striker. Using the expression

$$f(\xi) = -\frac{2n+1}{n+1} \cdot \frac{R}{t_0} \cdot \frac{d\xi}{d\tau},$$

from (9) and (14) we obtain an equation for the temperature maximum:

$$\frac{d\theta}{d\tau} = \left(-x \frac{d\xi}{d\tau} \right)^{n+1},$$

$$\theta = T/T_0, \quad T_0 = \frac{2^{n+1} m}{\rho_0 c_p} \left(\frac{2n+1}{n}\right)^{n+1} \frac{|w_0|^n R^{n+1}}{\delta_0^{2n+1}}. \tag{15}$$

It can be shown that the solution of the system of equations (12), (13), (15) reduces to the classical Cauchy problem for an autonomous system of seven first-order equations with initial conditions

$$x(0) = 1, \quad y(0) = 1, \quad \xi(0) = b, \quad a(0) = a_0, \quad \theta(0) = 0,$$

$$\Pi(0) = b^2 P_a + 2\int_b^1 P_0 \rho d\rho; \quad P_c = P_a - \int_b^\rho |\psi_0|^{n-1} \psi_0 d\rho,$$

$$\psi_0 = \rho - a_0^2/\rho, \quad \dot{y}(0) = -\beta\Pi(0),$$

$$\int_{a_0}^1 |\psi_0|^{n-1} \psi_0 d\rho + \int_b^{a_0} |\psi_0|^{n-1} \psi_0 d\rho = 0.$$

The difficulty of solving the system (12), (13), (15) analytically lies primarily in the impossibility of computing the integrals of ψ^n for arbitrary n. In special cases, for example, n = 0 and n = 1, they are readily computed, and exact solutions exist if additional assumptions are applied to the impact parameters.

For n = 1 we have m = μ, and from (11) we determine the pressure distribution [4]:

$$P = P_a \left(\frac{b^2 x}{\xi^2}\right)^\gamma - g\left(\frac{\rho^2 - \xi^2}{2} - a^2 \ln \frac{\rho}{\xi}\right).$$

If the striker is absolutely nondeformable (α = 0) and has a mass much greater than the mass of the fluid (β = 0), then for zero counterpressure of the gas in the cavity (P_a = 0) Eqs. (12), (13), and (15) have the solution

$$\begin{gathered} y = 1, \quad x = (1-\tau)^{-1}, \quad \tau = 1 - A(1 - \xi^2 + 2\xi^2 \ln \xi)^{-1}, \\ a^2 = -\frac{1-\xi^2}{2\ln \xi}; \quad \theta = \frac{1}{4A^3} \int_{b^2}^{\xi^2} \frac{(1 - q + q\ln q)^4}{q \ln q} dq < \frac{1}{4A^3} \ln \frac{\ln \xi}{\ln b}; \\ A = 1 - b^2 + 2b^2 \ln b, \\ \Pi = (1-\xi^2)(1+\xi^2 - 2a^2)x^3/4. \end{gathered} \tag{16}$$

It follows from (16) that the collapse time of the cavity is finite: $\tau_c = 1 - A$. For $\xi \to 0$ we have $x_c = A^{-1}$, a_c = 0, $\Pi_c = 1/4A^3$, and

$$\theta_c \sim \ln |\ln \xi| \to \infty.$$

For n = 0 we have m = k, and from (11) we determine the pressure profile:

$$P = \begin{cases} P_b + g(\rho - \xi) & \text{for } \xi < \rho < a, \\ P_b + g(2a - \xi - \rho) & \text{for } a < \rho < 1, \end{cases}$$

$$a = \frac{1+\xi}{2} - \frac{P_a}{x}\left[\left(\frac{b^2 x}{\xi^2}\right)^\gamma - 1\right].$$

For $\alpha = \beta = P_a = 0$ we find the flow parameters

$$\begin{gathered} y = 1, \quad x = (1-\tau)^{-1}, \quad \tau = 1 - B(1-\xi)^{-2}(1+3\xi)^{-2/3}, \\ a = (1+\xi)/2, \quad \theta = B\int_\xi^b (1-q)^2 (1+3q)^{2/3} dq, \\ B = (1-b)^2(1+3b)^{2/3}, \quad \Pi = x(1+\xi^3 - 2a^3)/3. \end{gathered} \tag{17}$$

It follows from (17) that for $\xi \to 0$ the time $\tau_c = 1 - B$ and

$$x_c = B^{-1}, \quad a_c = 1/2, \quad \Pi_c = 1/4B,$$

$$\theta_c = B\left\{(1+3b)^{5/3}\left[\frac{16}{45} - \frac{3}{24}(1+3b) + \frac{1}{99}(1+3b)^2\right] - 0.4705\right\}.$$

The cavity collapse curves are given in Fig. 1 in coordinates (ξ, τ) for the case $\alpha = \beta = P_a = 0$ with n = 1 (curve 1) and n = 0 (curve 2) for b = 0.5. For all 0 < n < 1 the curves $\xi(\tau)$ are obviously situated between the indicated curves.

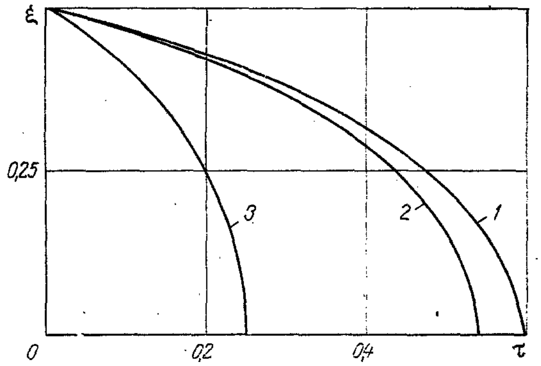

Fig. 1. Cavity radius versus impact time.

We can analyze the dynamics of collapse of the cavity in the absence of outward squeeze flow of the fluid, i.e., when $u(R, \delta) = 0$. Equations (12), (13), and (15) are applicable to this case if we put $a = 1$.

In particular, for $n = 1$ and $\alpha = \beta = P_a = 0$ we find

$$\tau = 1 - D(1-\xi^2)^{-1} = 1 - x^{-1}, \quad D = 1 - b^2,$$

$$\theta = \frac{1}{4D^3}\int_{\xi^2}^{b^2} \frac{(1-q)^4}{q}\,dq < \frac{1}{4D^3}\ln\frac{b}{\xi}, \tag{18}$$

$$\Pi = x^3[\xi^2(1-\xi^2/4) - \ln\xi - 3/4].$$

In the case $n = 0$

$$\tau = 1 - D(1-\xi^2)^{-1} = 1 - x^{-1},$$

$$\theta = D^{-1}[b(1-b^2/3) - \xi(1-\xi^2/3)], \tag{19}$$

$$\Pi = x[2/3 - \xi(1-\xi^2/3)].$$

It follows from (18) and (19) that the collapse curves $\xi(\tau)$ coincide for $n = 0$ and $n = 1$. Clearly, the same is true for all other $0 < n < 1$. The curve $\xi(\tau)$ for $b = 0.5$ is given in Fig. 1 (curve 3). Note the appreciable reduction in the collapse time of the cavity in transition from free flow (allowing outward squeeze flow of the fluid) to closed flow (outward flow excluded).

NOTATION

r, z, cylindrical coordinates; u, v, radial and axial components of fluid velocity; p, pressure; p_a, external (atmospheric) pressure; p_b, gas pressure in cavity; $\langle p \rangle$, average pressure over striker radius; R, striker radius; r_0, initial cavity radius; r_+, instantaneous cavity radius; $\dot{r}_+$, velocity of cavity wall; δ, thickness of fluid film; r_*, coordinate of neutral line; w, velocity of striker contact surface; t, time; m, n, rheological constants of fluid; μ, dynamic viscosity of fluid; k, yield point of solid in pure shear; γ, adiabatic exponent of gas; ρ_0, fluid density; c_p, specific heat at constant pressure; λ, thermal conductivity; T, absolute temperature; s_{ij}, stress deviator; d_{ij}, strain-rate tensor; τ_{rz}, shear stress; μ_a, effective viscosity of fluid; K, stiffness of impact system elements; c, constant of integration in Eq. (9).

LITERATURE CITED

1. A. V. Dubovik and V. K. Bobolev, Inzh.-Fiz. Zh., 27, No. 2, 317 (1974).
2. M. J. Booth and W. Hirst, Proc. Roy. Soc. (London), Ser. A, 316, 391; 451 (1970).
3. W. L. Wilkinson, Non-Newtonian Fluids, Pergamon, New York (1960).
4. É. I. Andriankin, V. K. Bobolev, and A. V. Dubovik, Zh. Prikl. Mekh. Tekh. Fiz., No. 6, 98 (1970).